Model Reduction by Balanced Truncation of Linear Systems with a Quadratic Output

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Abstract. Balanced truncation is a widely used and appreciated projection-based model reduction technique for linear systems. This technique has the following two important properties: approximations by balanced truncation preserve the stability and the $\mathscr{H}_\infty$-norm (the maximum of the frequency response) of the error system is bounded above by twice the sum of the neglected singular values. This paper tries to extend the framework of linear balanced truncation to systems with a quadratic output. For such systems, the controllability Gramian remains the same. The observability Gramian is computed from a linear system with multiple outputs that is derived from the quadratic output of the original system. We give a numerical example for a large-scale system arising from structural analysis.

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INTRODUCTION

The goal is to adapt the framework of linear balanced truncation to systems **Q** with a quadratic output

$$\mathbf{Q} = \begin{cases} s^2Mq(s) + sDq(s) + Kq(s) &= Bu(s) \\ q(s)^TSq(s) &= y(s) \end{cases}, \tag{1}$$

where K, D and $M \in \mathbb{R}^{n\times n}$ are respectively the stiffness matrix, the damping matrix and the non-singular mass matrix, $B \in \mathbb{R}^n$ and $S \in \mathbb{R}^{n\times n}$ is non-symmetric and of rank r with $r \ll n$, $q(s) \in \mathbb{R}^n$ is the state vector, $u(s) \in \mathbb{R}$ is the control input and $y(s) \in \mathbb{R}$ is the output. Note that $s = i\omega$ where ω is the frequency. Systems of the form **Q** have two difficulties which make it impossible to use the standard balanced truncation method: first, the second order state equation and, second, the non-linear output equation.

Methods and approaches for structure preserving model reduction of second order systems using balanced truncation are proposed in the literature [1, 2, 3, 4, 5]. In this paper, the stress is on the quadratic output and less on the efficient reduction of second order systems.

Since our approach can be combined with any method for the model reduction of second order systems with linear output, we chose in this paper, as a proof of concept of the method, for a classical model reduction approach for second order systems where we first linearize the second order state equation. By linearization, we mean the transformation of the state equation of (1), which is linear and of second order, into a set of two coupled linear equations of first order [7, 8]. We rewrite (1) as a first order generalized state space system with an output which is still quadratic

$$\mathbf{Q}_{\text{lin}} = \begin{cases} s\mathscr{E}x(s) &= \mathscr{A}x(s) + \mathscr{B}u(s) \\ y(s) &= x(s)^T\mathscr{S}x(s) \end{cases}, \tag{2}$$

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where $x(s) = \left[q(s)^T, sq(s)^T\right]^T$. Note that by this linearization, the dimension of the system doubles: $x(s) \in \mathbb{R}^{2n}$, $\mathscr{E} \in \mathbb{R}^{2n\times 2n}$, $\mathscr{A} \in \mathbb{R}^{2n\times 2n}$, $\mathscr{B} \in \mathbb{R}^{2n}$ and $\mathscr{S} \in \mathbb{R}^{2n\times 2n}$. An example of linearization is

$$\mathscr{E} = \begin{bmatrix} C & M \\ I & 0 \end{bmatrix} \quad , \quad \mathscr{A} = \begin{bmatrix} -K & 0 \\ 0 & I \end{bmatrix} \quad , \quad \mathscr{S} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathscr{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

A reduced model $\widehat{\mathbf{Q}}$ is obtained by first choosing matrices W, $V \in \mathbb{R}^{2n\times k}$ each with orthonormal columns and then defining

$$\widehat{\mathbf{Q}} = \begin{cases} s\mathscr{E}_r\hat{x}(s) &= \mathscr{A}_r\hat{x}(s) + \mathscr{B}_r u(s) \\ \hat{y}(s) &= \hat{x}(s)^T\mathscr{S}_r\hat{x}(s) \end{cases} , \tag{3}$$

where $\mathscr{E}_r = W^T\mathscr{E}V$, $\mathscr{A}_r = W^T\mathscr{A}V$, $\mathscr{B}_r = W^T\mathscr{B}$ and $\mathscr{S}_r = V^T\mathscr{S}V$.

LINEAR BALANCED TRUNCATION

Balanced truncation [9] is a widely used model reduction technique for linear systems. This technique was first introduced by Mullis and Roberts [10] and later in systems and control theory by Moore [11]. The underlying approximation theory was developed by Glover [12]. Consider the linear-time invariant system

$$\mathbf{L} = \begin{cases} sx(s) &= Ax(s) + Bu(s) \\ y(s) &= Cx(s) \end{cases} , \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(s) \in \mathbb{R}^n$ is the state, $u(s) \in \mathbb{R}^m$ is the input and $y(s) \in \mathbb{R}^p$ is the output. The number of states n is called the dimension or order of the system $\mathbf{L}$. Closely related to this system are two Lyapunov equations:

$$A\mathscr{P} + \mathscr{P}A^T + BB^T = 0, \qquad A^T\mathscr{Q} + \mathscr{Q}A + C^TC = 0 \tag{5}$$

where $\mathscr{P} \in \mathbb{R}^{n\times n}$ and $\mathscr{Q} \in \mathbb{R}^{n\times n}$ are respectively called the controllability and observability Gramians. These matrices are unique, symmetric and positive definite under the assumption that A is stable, i.e. all eigenvalues of A have negative real parts, and (4) is controllable and observable. These Gramians $\mathscr{P}$ and $\mathscr{Q}$ play a central role in balanced truncation, because the balanced truncation technique chooses the columns of the matrices W and V as the left and right eigenvectors associated with the k largest eigenvalues of the product $\mathscr{P}\cdot\mathscr{Q}$. The square roots of the eigenvalues of $\mathscr{P}\cdot\mathscr{Q}$ are the singular values of the Hankel operator associated with $\mathbf{L}$ and are called the Hankel singular values σ_i of the system $\mathbf{L}$:

$$\sigma_i(\mathbf{L}) = \sqrt{\lambda_i(\mathscr{P}\mathscr{Q})}.$$

Theorem 1 *Given the controllable, observable, stable and balanced system* $\tilde{\mathbf{L}}$*, the reduced-order system* $\hat{\mathbf{L}}$ *obtained by balanced truncation is again stable and the* $\mathscr{H}_\infty$*-norm (the maximum of the frequency response) of the difference between the full-order system* $\mathbf{L}$ *and the reduced-order system* $\hat{\mathbf{L}}$ *is upper bounded by twice the sum of the neglected Hankel singular values, multiplicities not included:*

$$\|y - \hat{y}\|_{\mathscr{H}_\infty} \leq 2(\sigma_{p+1} + \cdots + \sigma_q). \tag{6}$$

The above theorem states that if the neglected Hankel singular values are small, then the systems $\mathbf{L}$ and $\hat{\mathbf{L}}$ are guaranteed to be close. Note that (6) is an a priori error bound. Hence, given an error tolerance, we can decide how many states we have to truncate without forming the reduced model.

BALANCED TRUNCATION OF SYSTEMS WITH A QUADRATIC OUTPUT

In this section, we extend the framework of linear balanced truncation to system $\mathbf{Q}$ (1) with a quadratic output. Therefore we use the linearized form $\mathbf{Q}_{\text{lin}}$ (2).

As discussed higher, the Gramians play an important role in balanced truncation. After the linearization of $\mathbf{Q}$, it is possible to readily compute the controllability Gramian $\mathscr{P}$ of the linearized system $\mathbf{Q}_{\text{lin}}$ from

$$\mathscr{A}\mathscr{P}\mathscr{E}^T + \mathscr{E}\mathscr{P}\mathscr{A}^T + \mathscr{B}\mathscr{B}^T = 0.$$

Note that also the size of this Gramian is doubled ($\mathscr{P} \in \mathbb{R}^{2n\times 2n}$). This controllability Gramian $\mathscr{P}$ is still a symmetric positive definite matrix. This is not possible any longer for the observability Gramian $\mathscr{Q}$.

We will now give an alternative to obtain a 'Gramian' $\mathscr{Q}$ from the non-linear output equation of (2). In this method, we use the symmetric part of the matrix $\mathscr{S}$. Next, we use $\mathscr{P}$ and $\mathscr{Q}$ to form the matrices W and V and we apply balanced truncation to the complete system $\mathbf{Q}$.

As the output $y(s)$ of system $\mathbf{Q}_{\text{lin}}$ is scalar, it is easy to see that

$$y(s) = x(s)^T \frac{\mathscr{S}+\mathscr{S}^T}{2} x(s).$$

First, assume that $(\mathscr{S}+\mathscr{S}^T)/2$ is symmetric positive definite. Then there is an $\mathscr{L}$ so that $\mathscr{L}\mathscr{L}^T = (\mathscr{S}+\mathscr{S}^T)/2$ with $\mathscr{L} \in \mathbb{R}^{2n\times r}$. The output can then be written as $y(s) = \|z(s)\|_2^2$ where $z \in \mathbb{R}^r$ is the output of

$$\mathbf{Q}_2 = \begin{cases} s\mathscr{E}x(s) &= \mathscr{A}x(s) + \mathscr{B}u(s) \\ z(s) &= \mathscr{L}^T x(s) \end{cases}.$$

Now, this is a linear system with single input and multiple outputs (SIMO). A reduced model $\widehat{\mathbf{Q}}_2$ is obtained using a method for linear systems. The generalized Laypunov equation for the observability Gramian $\mathscr{Q}_2$ is

$$\mathscr{A}^T\mathscr{Q}_2\mathscr{E} + \mathscr{E}^T\mathscr{Q}_2\mathscr{A} + \frac{1}{2}(\mathscr{S}+\mathscr{S}^T) = 0,$$

since $(\mathscr{S}+\mathscr{S}^T)/2 = \mathscr{L}\mathscr{L}^T$. We have the following connection with the error on $y(s) = \|z(s)\|_2^2$: when $\|z(s)\|_2$ is bounded on the $i\omega$ axis, we have that

$$\max_{\omega\in\mathbb{R}} \left| \|z(i\omega)\|_2^2 - \|\hat{z}(i\omega)\|_2^2 \right| \leq \max_{\omega\in\mathbb{R}} \|z(i\omega) - \hat{z}(i\omega)\|_2 \cdot \|z(i\omega) + \hat{z}(i\omega)\|_2.$$

The factor $\|z(i\omega) - \hat{z}(i\omega)\|_2$ is bounded by the truncation error obtained from balanced truncation of system $\mathbf{Q}_2$.

When $\mathscr{S}+\mathscr{S}^T$ is indefinite, we form the factorization $(\mathscr{S}+\mathscr{S}^T)/2 = \mathscr{L}D\mathscr{L}^T$ with D a diagonal matrix with ± 1 on the main diagonal. We still develop a reduced model for $\mathbf{Q}_2$ and compute $\hat{y}(s)$ as $\hat{z}(s)^T D\hat{z}(s)$.

Using the controllability Gramian $\mathscr{P}$ and the 'Gramian' $\mathscr{Q}_2$ we can form the matrices W and V and we obtain the following reduced order model $\widehat{\mathbf{Q}}$

$$\widehat{\mathbf{Q}} = \begin{cases} sW^T\mathscr{E}V\hat{x}(s) &= W^T\mathscr{A}V\hat{x}(s) + W^T\mathscr{B}u(s) \\ \hat{y}(s) &= \hat{x}(s)^T V^T\mathscr{S}V\hat{x}(s) \end{cases} \quad \text{or} \quad \widehat{\mathbf{Q}} = \begin{cases} s\mathscr{E}_r\hat{x}(s) &= \mathscr{A}_r\hat{x}(s) + \mathscr{B}_r u(s) \\ \hat{y}(s) &= \hat{x}(s)^T\mathscr{S}_r\hat{x}(s) \end{cases}$$

where $\mathscr{E}_r \in \mathbb{R}^{k\times k}$, $\mathscr{A}_r \in \mathbb{R}^{k\times k}$, $\mathscr{B}_r \in \mathbb{R}^k$, $\mathscr{S}_r \in \mathbb{R}^{k\times k}$, $\hat{x}(s) \in \mathbb{R}^k$, $u(s) \in \mathbb{R}$ and $\hat{y}(s) \in \mathbb{R}$. We can prove that this reduced order system $\widehat{\mathbf{Q}}$ is again stable.

NUMERICAL EXPERIMENT

This section shows the results of balanced truncation applied on an application with a model with a quadratic output. In this application, we consider the model of a floor inside a building [13] with dimensions 10m × 10m × 0.3m. Its Young's modulus, Poisson's ratio, proportional damping ratio and density are respectively 30GPa, 0.3, 0.1 and 2500 kg/m^3. We use a DKT shell element finite element model [14] for the floor. The excitation is a point load in the middle point of the floor. This leads to matrices of order 29799. The matrix S computes the square mean of the displacement in four points selected around the point of excitation. This leads to a positive semidefinite matrix S of rank four. We used a block form of the Arnoldi-Lyapunov method [15]. See [6] for a structure preserving implementation for this type of problems.

The Gramians were approximated by matrices of rank 16. Figure 1 shows the output of the model for a unit input and the errors of the reduced output for different orders of the reduced models. Evaluating the large-scale model in 201 frequencies $\omega = 0:1:200$, takes in MATLAB about 6 min and 15 sec. For an increasing number of frequencies, the computation time scales linearly. On the other hand, the evaluation of the reduced order model is very fast, but we also have to take the computation of the reduced order model into account. The construction of the Gramians dominates the computation time, but the total computation time to calculate the output of the reduced model of order $k = 16$ takes only 45 sec in MATLAB. The extra computation time for an increasing number of frequencies is almost negligible. Using model reduction by balanced truncation on this system gives a gain factor of the computation time of more than 8. This illustrates the importance of balanced truncation on systems with a quadratic output.

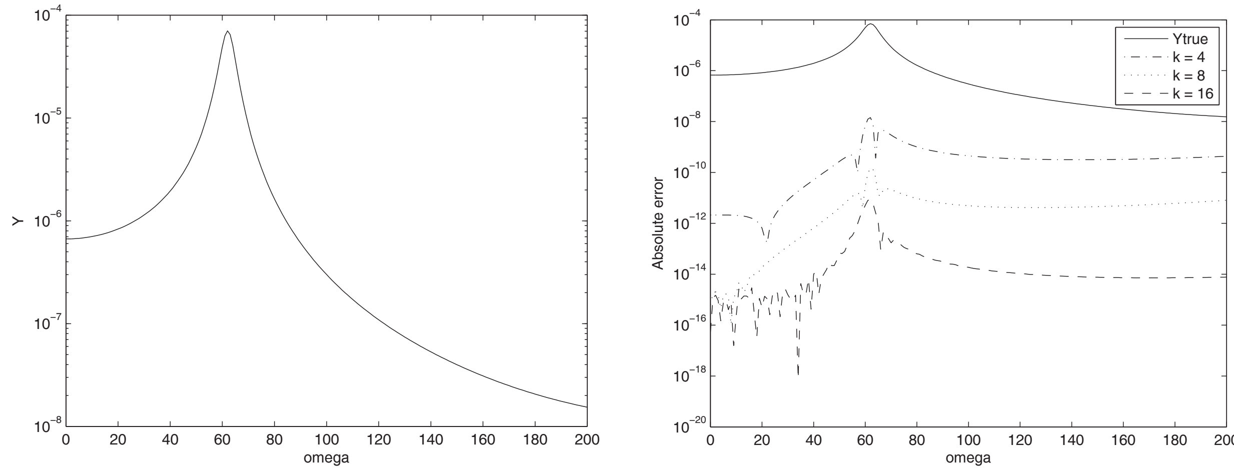

FIGURE 1. The results of the numerical experiments: (left) the output of the model for a unit input and (right) the error of the error of the reduced outputs for a unit input and order $k = 4, 8, 16$.

CONCLUSIONS

The framework of balanced truncation is extended to linear systems with a quadratic output. The proposed method writes the SISO system with a quadratic output as a linear SIMO system and uses this system to construct the matrices V and W. The numerical experiments on a practical application show that using the proposed method of balanced truncation on systems with a quadratic output can give a significant time gain.

REFERENCES

1. Y. Chahlaoui, K. A. Gallivan, A. Vandendorpe, and P. Van Dooren, “Model Reduction of Second-Order System,” in *Dimension Reduction of Large-Scale Systems*, edited by P. Benner, V. Mehrmann, and D. Sorensen, Springer-Verlag, Berlin/Heidelberg, Germany, 2005, vol. 45 of *Lecture Notes in Computational Science and Engineering*, pp. 149–172.
2. Y. Chahlaoui, D. Lemonnier, K. Meerbergen, A. Vandendorpe, and P. Van Dooren, “Model reduction of second order systems,” in *Proceedings International Symposium Mathematical Theory of Networks and Systems*, 2002.
3. T. Reis, and T. Stykel, *Mathematical and Computer Modelling of Dynamical Systems: Methods, Tools and Applications in Engineering and Related Sciences* **14**, 391–406 (2008).
4. T. Stykel, “Balanced truncation model reduction of second-order systems,” in *Proceedings of the 5th Vienna Symposium on Mathematical Modelling (5th MATHMOD, Vienna, Austria, February 8-10, 2006)*, edited by I. Troch, and F. Breitenecker, ARGESIM-Verlag, Vienna, 2006, vol. 2.
5. J. Saak, *Efficient Numerical Solution of Large Scale Algebraic Matrix Equations in PDE Control and Model Order Reduction*, Phdthesis, Universty of Chemnitz (2009).
6. R. Van Beeumen, *Model reduction by balanced truncation of linear systems with a quadratic output*, Master thesis, K.U.Leuven (2010), in Dutch.
7. A. Amiraslani, R. M. Corless, and P. Lancaster, *IMA Journal of Numerical Analysis* **29**, 141–157 (2009).
8. D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *SIAM Journal on Matrix Analysis and Applications* **28**, 971–1004 (2006).
9. A. C. Antoulas, *Approximation of large-scale dynamical systems (Advances in Design and Control)*, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2005.
10. C. T. Mullis, and R. A. Roberts, *IEEE Transactions on Circuits and Systems* pp. 551–561 (1976).
11. B. C. Moore, *IEEE Transactions on Automatic Control* **26**, 17–32 (1981).
12. K. Glover, *International Journal of Control* **39**, 1115–1193 (1984).
13. Y. Yue, and K. Meerbergen, Using model order reduction for the design optimization of structures and vibrations, Technical Report TW566, Department of Computer Science, K.U.Leuven, Celestijnenenlaan 200A, 3001 Heverlee (Leuven), Belgium (2010), submitted for publication.
14. C. R. Calladine, *Theory of shell structures*, Cambridge University Press, 1989.
15. I. M. Jaimoukha, and E. M. Kasenally, *SIAM Journal on Numerical Analysis* **31**, 227–251 (1994).